Chapter 5

Solving Initial-Value Problem for ODE

Initial value problem： $\dfrac{dy}{dt} = f(t, y),\ a \le t \le b,\ y(a) = \alpha.$

1. Single step methods：

Runge-Kutta methods：

（1） Euler's method：

$w_0 = \alpha$,

$w_{i+1} = w_i + hf(t_i, w_i)$, for each $i = 1,2,\ldots,N$.

（2） Midpoint method：

$w_0 = \alpha$,

$w_{i+1} = w_i + hf\left(t_i + \dfrac{h}{2}, w_i + \dfrac{h}{2} f(t_i, w_i)\right)$, for each

$i = 0,1,\ldots,N-1$.

（3） Modified Euler method：

$w_0 = \alpha$,

$w_{i+1} = w_i + \dfrac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$, for each

$i = 0,1,\ldots,N-1$.

（4） Heun's method：

$w_0 = \alpha$,

$w_{i+1} = w_i + \dfrac{h}{4}[f(t_i, w_i) + 3f(t_i + \dfrac{2}{3}h, w_i + \dfrac{2}{3}hf(t_i, w_i))]$, for each $i = 0,1,\ldots,N-1$.

（5） Classic Runge-Kutta method：

$$w_0 = \alpha,$$

$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1),$$

$$k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2),$$

$$k_4 = hf(t_{i+1}, w_i + k_3),$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$ for each $i = 0,1,\ldots,N-1$.

（6） Runge-Kutta Fehlberg method：

This technique consists of using a Runge-Kutta method with local truncation error of order five,

$$\widetilde{w}_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6,$$

to estimate the local error in a Runge-Kutta method of order four,

$$w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5,$$

where

$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf\left(t_i + \frac{h}{4}, w_i + \frac{1}{4}k_1\right),$$

$$k_3 = hf\left(t_i + \frac{3h}{8}, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right),$$

$$k_4 = hf\left(t_i + \frac{12h}{13}, w_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right),$$

$$k_5 = hf\left(t_i + h, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right),$$

$$k_6 = hf\left(t_i + \frac{h}{2}, w_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right).$$

An advantage to this method is that only six evaluations of f are required per step.

2. Linear multistep methods：

Adams-Bashforth methods：

（1） One-step: Euler's method.

（2） Two-step:

$$w_0 = \alpha,\ w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{2}[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})],$$ for each

$i = 1,2,\ldots, N-1.$

（3） Three-step:

$$w_0 = \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})],$$

for each $i = 2,3,\ldots, N-1.$

（4） Four-step:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$

$$w_{i+1} = w_i + \frac{h}{12}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1})$$

$$+ 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})],$$

for each $i = 3,4,\ldots, N-1$.

Adams-Moulton methods：

（1） One-step: Trapezoidal method,

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)],$$ for each

$i = 0,1,\ldots, N-1$.

（2） Two-step:

$$w_0 = \alpha, \; w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})],$$

for each $i = 1,2,\ldots, N-1$.

（3） Three-step:

$$w_0 = \alpha, \; w_1 = \alpha_1, \; w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i)$$

$$-5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})],$$ for each

$i = 2,3,\ldots, N-1$.

Adams Fourth-Order Predictor-Corrector：

$$\hat{w}_{i+1} = w_i + \frac{h}{12}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1})$$

$$+37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})],$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, \hat{w}_{i+1}) + 19f(t_i, w_i)$$

$$-5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})].$$

Adams Variable-Step-Size Fourth-Order Predictor-Corrector：

$$w_{i+1}^{(0)} = w_i + \frac{h}{12}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1})$$

$$+37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})],$$

$$w_{i+1} = w_i + \frac{h}{24}\left[9f(t_{i+1}, w_{i+1}^{(0)}) + 19f(t_i, w_i)\right.$$

$$\left.-5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right],$$

with an estimate of the local truncation error of the corrector step:

$$|\tau_{i+1}(h)| = \left|\frac{y(t_{i+1}) - w_{i+1}}{h}\right| \approx \frac{19h^4}{720} \cdot \frac{8}{3h^5}\left|w_{i+1} - w_{i+1}^{(0)}\right|$$

$$= \frac{19\left|w_{i+1} - w_{i+1}^{(0)}\right|}{270h},$$

followed by a process of choosing h and a criterion of changing h when necessary. A change in step size for a multistep method is more costly in terms of function evaluations than for a one-step method, since new equally spaced starting values must be computed. So we tend to do this conservatively.

Milne's method：

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})].$$

Simpson's method：

$$w_{i+1} = w_{i-1} + \frac{h}{3}[f(t_{i+1}, w_{i+1}) + 4f(t_i, w_i) + f(t_{i-1}, w_{i-1})].$$

Milne-Simpson Predictor-Corrector：

$$\hat{w}_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})],$$

$$w_{i+1} = w_{i-1} + \frac{h}{3}[f(t_{i+1}, \hat{w}_{i+1}) + 4f(t_i, w_i) + f(t_{i-1}, w_{i-1})].$$

Note: However, Milne, Simpson, Milne-Simpson Predictor-Corrector methods are all inherently A-unstable and therefore are of limited use.

BDF（Backward Differential Formula）：

（1） One-step: implicit Euler's method,

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}).$$

（2） Two-step:

$$w_0 = \alpha,\ w_1 = \alpha_1,$$

$$w_{i+1} = \frac{4}{3}w_i - \frac{1}{3}w_{i-1} + \frac{2}{3}hf(t_{i+1}, w_{i+1}).$$

（3） Three-step:

$$w_0 = \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,$$

$$w_{i+1} = \frac{18}{11}w_i - \frac{9}{11}w_{i-1} + \frac{2}{11}w_{i-2} + \frac{6}{11}hf(t_{i+1}, w_{i+1}).$$

（4） Four-step:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$

$$w_{i+1} = \frac{48}{25} w_i - \frac{36}{25} w_{i-1} + \frac{16}{25} w_{i-2} - \frac{3}{25} w_{i-3} + \frac{12}{25} hf(t_{i+1}, w_{i+1}).$$

（5） Five-step:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3, \ w_4 = \alpha_4,$$

$$w_{i+1} = \frac{300}{137} w_i - \frac{300}{137} w_{i-1} + \frac{200}{137} w_{i-2}$$

$$- \frac{75}{137} w_{i-3} + \frac{12}{137} w_{u-4} + \frac{60}{137} hf(t_{i+1}, w_{i+1}).$$

（6） Six-step:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3, \ w_4 = \alpha_4, \ w_5 = \alpha_5,$$

$$w_{i+1} = \frac{360}{147} w_i - \frac{450}{147} w_{i-1} + \frac{400}{147} w_{i-2} - \frac{225}{147} w_{i-3}$$

$$+ \frac{72}{147} w_{u-4} - \frac{10}{147} + \frac{60}{147} hf(t_{i+1}, w_{i+1}).$$

3. MATLAB ODE suite：

Non-Stiff：ode23, ode45, ode113.

Stiff：ode15s, ode23s.

For details read 【2】.

References:

【1】 R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS, Boston, 1993.

【2】 L. F. Shampine and M. W. Reichelt, “The Matlab ODE Suite,” *Siam Journal on Scientific Computing*, Volume 18, Number 1, pp1-22, 1997.