Chapter 8 Numerical Solutions of Nonlinear Systems of Equations

Steepest Descent:

The method of Steepest Descent determines a local minimum for a multivariable function of the form $g:\mathbf{R}^n \to \mathbf{R}$. The connection between the minimization of a function from $\mathbf{R}^n$ to $\mathbf{R}$ and the solution of a system of nonlinear equations is due to the fact that a system of the form

$$f_1(x_1, x_2, \ldots, x_n) = 0,$$
$$f_2(x_1, x_2, \ldots, x_n) = 0,$$
$$\vdots$$
$$f_n(x_1, x_2, \ldots, x_n) = 0,$$

has a solution at $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ precisely when the function g defined by

$$g(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} [f_i(x_1, x_2, \ldots, x_n]^2$$

has the minimal value zero.

The method of Steepest Descent for finding a local minimum for an arbitrary function g from $\mathbf{R}^n$ into $\mathbf{R}$ can be intuitively described as follows:

i. Evaluate g at an initial approximation $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^t$;

ii. Determine a direction from $\mathbf{x}^{(0)}$ that results in a decrease in the value of g;

iii. Move an appropriate distance in this direction and call the new vector $\mathbf{x}^{(1)}$;

iv. Repeat steps i through iii with $\mathbf{x}^{(0)}$ by $\mathbf{x}^{(1)}$.

An appropriate choice for $\mathbf{x}^{(1)}$ is

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - \alpha \nabla g(\mathbf{x}^{(0)}), \qquad \text{for some constant } \alpha > 0.$$

To determine an appropriate choice of α, we consider the single-variable function

$$h(\alpha) = g(\mathbf{x}^{(0)} - \alpha \nabla g(\mathbf{x}^{(0)})).$$

The value of α that minimize $h(\alpha)$ is what we need.

References:

【1】 R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS, Boston, 1993.