Chapter 9 Boundary-Value Problems for Ordinary Differential Equations

Boundary-Value Problems for ODE

Shooting method:

The shooting technique for the nonlinear second-order boundary-value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \; y(a) = \alpha, \; y(b) = \beta, \qquad (1)$$

is using the solutions to a sequence of initial-value problems of the form

$$y'' = f(x, y, y'), \quad a \le x \le b, \; y(a) = \alpha, \; y'(a) = t, \qquad (2)$$

involving a parameter t, to approximate the solution to our boundary-value problem. We do this by choosing the parameters $t = t_k$ in a manner to ensure that

$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta,$$

where $y(x, t_k)$ denotes the solution to the initial-value problem (2) with $t = t_k$ and $y(x)$ denotes the solution to the boundary-value problem (1). This technique is called a ‘‘shooting’’ method, by analogy to the procedure of firing objects at a stationary target. We start with a parameter t_0 that determines the initial elevation at which the object is fired from the point (a, α) and along the curve described by the solution to the initial-value problem:

$$y'' = f(x, y, y'), \quad a \le x \le b, \; y(a) = \alpha, \; y'(a) = t_0.$$

If $y(b, t_0)$ is not sufficiently close to β, we correct tour approximation by choosing elevations t_1, t_2, and so on, until $y(b, t_k)$ is sufficiently close to ''hitting'' β.

Finite difference method:

For the general nonlinear boundary-value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \; y(a) = \alpha, \; y(b) = \beta$$

to be solved by finite difference method, we first divide $[a, b]$ into $(N+1)$ equal subintervals whose endpoints are at $x_i = a + ih$ for $i = 0, 1, \ldots, N+1$. Assuming that the exact solution has a bounded fourth derivative allows us to replace $y'(x_i)$ and $y''(x_i)$ in each of the equations

$$y''(x_i) = f(x_i, y(x_i), y'(x_i))$$

by the appropriate centered-difference formula given below,

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12} y^{(4)}(\xi_i), \qquad (3)$$

$$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6} y'''(\eta_i), \qquad (4)$$

to obtain, for each $i = 1, 2, \ldots, N$,

$$\frac{y(x_{i+1})-2y(x_i)+y(x_{i-1})}{h^2}=f\left(x_i,y(x_i),\frac{y(x_{i+1})-y(x_{i-1})}{2h}-\frac{h^2}{6}y'''(\eta_i)\right)$$

$$+\frac{h^2}{12}y^{(4)}(\xi_i),$$

for some ξ_i and η_i in the interval (x_{i-1},x_{i+1}).

Then the difference method results when the error terms are deleted and the boundary conditions employed:

$$w_0=\alpha,\quad w_{N+1}=\beta,$$

and

$$-\frac{w_{i+1}-2w_i+w_{i-1}}{h^2}+f\left(x_i,w_i,\frac{w_{i+1}-w_{i-1}}{2h}\right)=0,$$

for each $i=1,2,\ldots,N$.

The $N\times N$ nonlinear system obtained form this method,

$$2w_1-w_2+h^2f\left(x_1,w_1,\frac{w_2-\alpha}{2h}\right)-\alpha=0,$$

$$-w_1+2w_2-w_3+h^2f\left(x_2,w_2,\frac{w_3-w_1}{2h}\right)=0,$$

$$\vdots$$

$$-w_{N-2}+2w_{N-1}-w_N+h^2f\left(x_{N-1},w_{N-1},\frac{w_N-w_{N-2}}{2h}\right)=0,$$

$$-w_{N-1}+2w_N+h^2f\left(x_N,w_N,\frac{\beta-w_{N-1}}{2h}\right)-\beta=0,$$

is then solved by Newton’s method. The Jacobian matrix involved is tridiagonal.

References:

【1】 R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS, Boston, 1993.