# Chapter 7　Approximation Theory

## Approximation by Orthogonal Polynomials and Fourier Series

**Continuous Fourier transform and inverse transform:**

$$f(x) \approx \frac{\tilde{a}_0}{2} + \sum_{k=1}^{m} \tilde{a}_k \cos kx + \sum_{k=1}^{m} \tilde{b}_k \sin kx, \qquad -\pi \le x \le \pi \text{，}$$

$$\tilde{a}_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx \, dx \text{，} \qquad \forall k = 0,1,\ldots,m$$

$$\tilde{b}_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \, dx, \qquad \forall k = 1,\ldots,m \text{。}$$

**Discrete Fourier transform and inverse transform:**

$$f(x) \approx \frac{\hat{a}_0}{2} + \hat{a}_n \cos nx + \sum_{k=1}^{m-1} (\hat{a}_k \cos kx + \hat{b}_k \sin kx), \quad -\pi \le x \le \pi,$$

$$x_j = -\pi + \left(\frac{j}{m}\right)\pi, \quad y_j = f(x_j), \qquad \forall j = 0,1,\ldots,2m-1 \text{，}$$

$$\hat{a}_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \cos kx_j, \qquad \forall k = 0,1,\ldots,m \text{，}$$

$$\hat{b}_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \sin kx_j, \qquad \forall k = 1,2,\ldots,m-1 \text{。}$$

**Continuous Fourier transform and inverse transform (in**

**complex form):**

$$f(x) \approx \sum_{k=-m}^{m-1} \tilde{c}_k e^{ikx}, \qquad 0 \le x \le 2\pi ,$$

$$\tilde{c}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx}, \qquad k = -m, \ldots, m-1.$$

**Discrete Fourier transform and inverse transform (in complex form):**

$$f(x) \approx \sum_{k=-m}^{m-1} \hat{c}_k e^{ikx}, \quad 0 \le x \le 2\pi ,$$

$$x_j = \left(\frac{j\pi}{m}\right), \; y_j = f(x_j), \; \forall j = 0,1,\ldots,2m-1 ,$$

$$\hat{c}_k = \frac{1}{2m} \sum_{j=0}^{2m-1} y_j \, e^{-ikx_j}, \quad \forall k = -m, \ldots, m-1.$$

There are two ways to compute the above $c_k$. One is by matrix multiplication with number of multiplication of $O(m^2)$, the other is by FFT with number of multiplication of $O(m \log_2 m)$.

**Continuous Legendre transform and inverse transform:**

$$f(x) \approx \sum_{k=0}^{n} \tilde{a}_k P_k(x), \qquad -1 \le x \le 1 ,$$

where $P_k(x)$ is the k-th Legendre polynomial.

$$\tilde{a}_k = (k + 1/2) \int_{-1}^{1} f(x) \, P_k(x) \, dx , \qquad \forall k = 0, \ldots, n.$$

**Discrete Legendre-Gauss transform and inverse transform:**

$$f(x) \approx \sum_{k=0}^{n} \hat{a}_k P_k(x), \quad -1 \le x \le 1,$$

$$\hat{a}_k = (k+1/2) \sum_{j=0}^{n} w_j P_k(x_j) f(x_j) ,$$

where $x_j,\ j = 0,\dots,n,$ are roots of $P_{n+1}(x)$, and

$$w_j = \frac{2}{(1-x_j^2)[P'_{n+1}(x_j)]^2}, \qquad j = 0,\dots,n.$$

**Discrete Legendre-Gauss-Radau transform and inverse transform:**

$$f(x) \approx \sum_{k=0}^{n} \hat{a}_k P_k(x), \quad -1 \le x \le 1,$$

$$\hat{a}_k = (k+1/2) \sum_{j=0}^{n} w_j P_k(x_j) f(x_j) ,$$

where $x_j,\ j = 0,\dots,n,$ are roots of $P_n(x) + P_{n+1}(x)$, and

$$w_j = \begin{cases} \dfrac{2}{(n+1)^2}, & (j = 0), \\ \dfrac{1-x_j}{(n+1)^2[P_n(x_j)]^2}, & (j \ge 1). \end{cases}$$

**Discrete Legendre-Gauss-Lobatto transform and inverse transform:**

$$f(x) \approx \sum_{k=0}^{n} \hat{a}_k P_k(x), \quad -1 \le x \le 1,$$

$$\hat{a}_k = \begin{cases} (k+1/2) \sum_{j=0}^{n} w_j P_k(x_j) f(x_j), & (k < n), \\ \frac{n}{2} \sum_{j=0}^{n} w_j P_k(x_j) f(x_j), & (k = n), \end{cases}$$

where $x_j$, $j = 0, \ldots, n$, are $-1$, roots of $P_n'(x)$, and 1.

$$w_j = \frac{2}{n(n+1)[P_n(x_j)]^2}, \quad j = 0, \ldots, n.$$

## Continuous Chebyshev transform and inverse transform:

$$f(x) \approx \sum_{k=0}^{n} \tilde{a}_k T_k(x), \qquad -1 \le x \le 1,$$

where $T_k(x) = \cos(k \cos^{-1}(x))$ is the k-th Chebyshev polynomial.

$$\tilde{a}_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx, \text{ and}$$

$$\tilde{a}_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx, \qquad \forall k = 1, \ldots, n.$$

## Discrete Chebyshev-Gauss transform and inverse transform:

$$f(x) \approx \sum_{k=0}^{n} \hat{a}_k T_k(x), \quad -1 \le x \le 1,$$

$$\hat{a}_k = \begin{cases} \frac{1}{\pi} \sum_{j=0}^{n} w_j T_k(x_j) f(x_j), & (k = 0), \\ \frac{2}{\pi} \sum_{j=0}^{n} w_j T_k(x_j) f(x_j), & (1 \le k \le n), \end{cases}$$

where $x_j = \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$, $j = 0,\ldots,n,$ and

$$w_j = \frac{\pi}{n+1}, \qquad j = 0,\ldots,n.$$

**Discrete Chebyshev-Gauss-Radau transform and inverse transform:**

$$f(x) \approx \sum_{k=0}^{n} \hat{a}_k T_k(x), \quad -1 \le x \le 1,$$

$$\hat{a}_k = \begin{cases} \frac{1}{\pi}\sum_{j=0}^{n} w_j T_k(x_j) f(x_j), & (k=0), \\ \frac{2}{\pi}\sum_{j=0}^{n} w_j T_k(x_j) f(x_j), & (1 \le k \le n), \end{cases}$$

where $x_j = -\cos\left(\frac{2j\pi}{2n+1}\right)$, $j = 0,\ldots,n,$ and

$$w_j = \begin{cases} \frac{\pi}{2n+1}, & (j=0), \\ \frac{2\pi}{2n+1}, & (j \ge 1). \end{cases}$$

**Discrete Chebyshev-Gauss-Lobatto transform and inverse transform:**

$$f(x) \approx \sum_{k=0}^{n} \hat{a}_k T_k(x), \quad -1 \le x \le 1,$$

$$\hat{a}_k = \begin{cases} \dfrac{1}{\pi} \sum_{j=0}^{n} w_j T_k(x_j) f(x_j), & (k = 0, n), \\ \dfrac{2}{\pi} \sum_{j=0}^{n} w_j T_k(x_j) f(x_j), & (1 \le k \le n-1), \end{cases}$$

where $x_j = \cos\left(\dfrac{j\pi}{n}\right)$, $j = 0, \ldots, n$, and

$$w_j = \begin{cases} \dfrac{\pi}{2n}, & (j = 0, n), \\ \dfrac{\pi}{n}, & (1 \le j \le n-1). \end{cases}$$

## Aliasing

The relation of $\tilde{c}_k$ and $\hat{c}_k$ in complex Fourier series is

$$\tilde{c}_k = \hat{c}_k + \sum_{\substack{l=-\infty \\ l \ne 0}}^{\infty} \hat{c}_{k+2ml}, \qquad k = -m, \ldots, m-1.$$

It shows that the k-th mode of the trigonometric interpolant of $f(x)$ depends not only on the k-th mode of $f(x)$, but also on all the modes of $f(x)$ which “alias” the k-th one on the discrete grid. The $(k + 2ml)$-th frequency aliases the k-th frequency on the grid; they are indistinguishable at the nodes since $e^{i(k+2ml)x_j} = e^{ikx_j}$, where $x_j = \left(\dfrac{j\pi}{m}\right)$, $j = 0, 1, \ldots, 2m-1$.

Similarly, for Chebyshev polynomial expansion, $\tilde{a}_k$ and $\hat{a}_k$ in Chebyshev-Gauss-Lobatto transform are related as:

$$\tilde{c}_k = \hat{c}_k + \sum_{\substack{j=2nl\pm k \\ j>n}} \hat{c}_j, \qquad k = 0,\ldots,n.$$

That means the $(2nl \pm k)$-th frequency aliases the k-th frequency on the grid; they are indistinguishable at the nodes since

$T_k(x_j) = T_{2nl\pm k}(x_j)$, where $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j = 0,1,\ldots,n$.

References:


【1】 R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS, Boston, 1993.

【2】 B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, 1996.

【3】 C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, 1988.