Chapter 3 Interpolation and Polynomial Approximation

Interpolation for Monotonically Increasing Discrete Nodes

Lagrange interpolation polynomial：

In the software, given $\{(x_k, y_k)\}$, $k = 1,2,\cdots,n+1$, with $x_1,\ldots,x_{n+1}$ being monotically increasing, then there exists a unique polynomial P of degree at most n with the property that

$$y_k = P(x_k), \qquad \text{for each } k = 1,\ldots,n+1.$$

This polynomial is give by

$$P(x) = y_1 L_{n,1}(x) + \cdots + y_n L_{n,n+1}(x) = \sum_{k=1}^{n+1} y_k L_{n,k}(x),$$

where

$$L_{n,k}(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_{n+1})}{(x_k-x_1)(x_k-x_2)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_{n+1})}$$

$$= \prod_{\substack{i=1 \\ i\neq k}}^{n+1} \frac{(x-x_i)}{(x_k-x_i)}, \text{ for each } k = 1,\ldots,n+1.$$

Hermite interpolation polynomial：

In the software, given $\{(x_k, y_k, y'_k)\}$, $k = 1,2,\cdots,n+1$, with $x_1, x_2,\ldots,x_{n+1}$ being monotically increasing, and y'_k being equal to the slope of the light blue dash bar associated with each (x_k, y_k), then there exists a unique polynomial H_{2n+1} of degree at most $2n+1$ agreeing with y and y' at $x_1,\ldots,x_{n+1}$, where

$$H_{2n+1}(x)=\sum_{j=1}^{n+1} y_j H_{n,j}(x)+\sum_{j=1}^{n+1} y'_j \hat{H}_{n,j}(x),$$

where $$H_{n,j}(x)=\left[1-2(x-x_j)L'_{n,j}(x_j)\right]L^2_{n,j}(x),$$

and $$\hat{H}_{n,j}(x)=(x-x_j)L^2_{n,j}(x).$$

$L_{n,j}$ denotes the j th Lagrange coefficient polynomail of degree n .

Natural cubic spline and Clamped cubic spline：

In the software, given $\{(x_k,y_k)\}, k=1,2,\cdots,n+1$, with $x_1,x_2,\ldots,x_{n+1}$ being monotically increasing, a cubic spline interpolant, S, for $\{(x_k,y_k)\}, k=1,2,\cdots,n+1$, is a function that satisfies the following conditions:

a. S is a cubic polynomial,denoted S_j, on the subinterval

$[x_j,x_{j+1}]$, where

$$S_j=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3,$$

$\forall j=1,2,\ldots,n+1$;

b. $S(x_j)=y_j, \quad \forall j=1,2,\ldots,n+1;$

c. $S_{j+1}(x_{j+1})=S_j(x_{j+1}), \quad \forall j=1,2,\ldots,n-1;$

d. $S'_{j+1}(x_{j+1})=S'_j(x_{j+1}), \quad \forall j=1,2,\ldots,n-1;$

e. $S''_{j+1}(x_{j+1})=S''_j(x_{j+1}), \quad \forall j=1,2,\ldots,n-1;$

f. One of the following set of boundary conditions is satisfied:

（i）$S''(x_1)=S''(x_{n+1})=0$ （free or nature boundary）;

（ii） $S'(x_1) = y_1'$ and $S'(x_{n+1}) = y_{n+1}'$ (clamped boundary），

where y_1' and y_{n+1}' are equal to the slope of the light blue dash bar associated with (x_1, y_1) and (x_{n+1}, y_{n+1}).

References:

【1】 R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS, Boston, 1993.