Chapter 3 Interpolation and Polynomial Approximation

Lagrange Interpolation

If $x_0, x_1, \ldots, x_n$ are $(n+1)$ distinct numbers and f is a function whose values are given at these numbers, then there exists a unique polynomial P of degree at most n with the property that

$$f(x_k) = P(x_k) \qquad \text{for each } k = 0,1,\ldots,n.$$

This Lagrange interpolation polynomial is give by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

where

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$, for each $k = 0,1,\ldots,n.$ If $f \in C^{n+1}[a,b]$, then for each $x \in [a,b]$, exists a number $\xi \in (a,b)$ with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)\,!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

$= P(x) + f^{(n+1)}(\xi(x))\,E_n(x),$ where $E_n(x)$ is particularly shown in the software to evaluate the quality of distribution of interpolation nodes.

In the software, several styles of distribution of interpolation nodes can be chosen. They are uniform distribution, Legendre-Gauss points, Legendre-Gauss-Radau points, Legendre-Gauss-Lobatto points, Chebyshev-Gauss points, Chebyshev-Gauss-Radau, and Chebyshev-Gauss-Lobatto points. Except the uniform distribution, the others are defined as below considering $[a,b]=[-1,1]$.

Let $P_n(x)$ be the Legendre polynomial of degree n, then
Legendre-Gauss points are the roots of $P_n(x)$;
Legendre-Gauss-Radau points are the roots of $P_n(x)+P_{n-1}(x)$;
Legendre-Gauss-Lobatto points are -1,1 and the roots of $P'_{n-1}(x)$.
Also, Chebyshev-Gauss points are $\cos\left(\frac{2k-1}{2n}\pi\right)$;
Chebyshev-Gauss-Radau points are $\cos\left(\frac{2(k-1)}{2n-1}\pi\right)$;
Chebyshev-Gauss-Lobatto points are $\cos\left(\frac{k}{n}\pi\right)$, for each $k=0,1,2,\ldots,n.$ These distributions of interpolation nodes related to Legendre and Chebyshev polynomials have special effects on equally distributing $E_n(x)$ over $[-1,1]$. For general $[a,b]$ other than $[-1,1]$, an additional linear mapping is required.

In the software, the default function $f(x)=\dfrac{1}{1+x^2}, \quad -5\le x\le 5,$ is called Runge function, for which its Lagrange polynomial based on uniform distribution fails to converge in $3.64<|x|<5$. However, this does not happen and full convergence on $-5\le x\le 5$ is recovered for the other distributions related to Legendre and Chebyshev polynomials.

References:

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【2】 K. E. Atkinson, *An Introduction to Numerical Analysis*, Wiley, Singapore, 1988.

【3】 B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, New York, 1996.