Chapter 3 Interpolation and Polynomial Approximation

Function Interpolation by Polynomial or Piecewise Polynomial

Lagrange interpolating polynomial :

If $x_0, x_1, \ldots, x_n$ are $(n+1)$ distinct numbers and f is a function whose values are given at these numbers, then there exists a unique polynomial P of degree at most n with the property that

$$f(x_k) = P(x_k) \qquad \text{for each } k = 0,1,\ldots,n.$$

This polynomial is give by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

where

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}, \text{ for each } k = 0,1,\ldots,n.$$

Natural and clamped cubic splines：

Given a function f defined on $[a,b]$, and a set of numbers, called nodes, $a = x_0 < x_1 < \cdots < x_n = b$, a cubic spline interpolant, S, for f is a function that satisfies the following conditions:

a. S is a cubic polynomial, denoted S_j, on the subinterval $[x_j, x_{j+1}]$, where

$S_j = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$,

$\forall j = 0,1,\ldots,n-1$;

b. $S(x_j) = f(x_j), \quad \forall j = 0,1,\ldots,n$;

c. $S_{j+1}(x_{j+1}) = S_j(x_{j+1}), \quad \forall j = 0,1,\ldots,n-2$;

d. $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}), \quad \forall j = 0,1,\ldots,n-2$;

e. $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}), \quad \forall j = 0,1,\ldots,n-2$;

f. One of the following set of boundary conditions is satisfied:

（i） $S''(x_0) = S''(x_n) = 0$ （free or nature boundary）;

（ii） $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ （clamped boundary）.

References:

【1】 R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS, Boston, 1993.